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An iterative boundary element method for solving the one-dimensional backward heat conduction problem

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Abstract

In this paper, the iterative algorithm proposed by V.A. Kozlov and V.G. Maz'ya [Leningrad Math. J. 5 (1990) 1207–1228] is numerically implemented using the boundary element method (BEM) in order to solve the backward heat conduction problem (BHCP). The convergence and the stability of the numerical method are investigated and a stopping criterion is proposed. The numerical results obtained confirm that the iterative BEM produces a convergent and stable numerical solution with respect to increasing the number of boundary elements and decreasing the amount of noise added into the input data.

1. Introduction

The transient heat conduction phenomena are generally governed by the parabolic heat conduction equation and if the initial temperature distribution and boundary conditions are specified, then this, in general, leads to a well-posed problem which may easily be solved numerically by using various numerical methods.

However, in many practical situations, when dealing with a heat conduction body it is not always possible to specify the boundary conditions or the initial temperature. For example, in practice, one may have to investigate the temperature distribution and the heat flux history from the known data at a particular time. In other words, it may be possible to specify the temperature distribution at a particular time, say $t = t_f > 0$ and either the temperature T or the heat flux $\partial T/\partial v$ on the boundary of the domain, and from this data the question arises as to whether the temperature distribution at any earlier time $t < t_f$ can be retrieved. This is usually referred to as the backward heat conduction problem (BHCP), or the final boundary value problem.

The systematic study of the BHCP is of rather recent origin, although consideration has already been given to such problems for several hundred years. In general, no solution which satisfies the heat conduction equation, the final data and the boundary conditions exists. Further, even if a solution did exist, it would not be continuously dependent on the boundary and the final data, see [7]. Thus, the BHCP is an example of an ill-posed problem that is impossible to solve using classical numerical methods and requires special techniques to be employed, see [1]. Conditions for which the BHCP becomes well posed have been investigated by Miranker [2] and Cannon and Douglas [4]. These studies introduced additional hypotheses which restrict the class of functions to which the solution must belong, and which are seldom satisfied. Therefore, numerical methods of solution appear more useful. Thus, regularization techniques, see [3,10], have been developed for solving the BHCP. Different methods, based on a perturbation of the original parabolic heat equation were proposed by Lattes and Lions [5] and Lesnic et al. [11].

In this study, we iteratively use the boundary element method (BEM) in order to implement a convergent algorithm which was first proposed by Kozlov and Maz'ya [9] and consists of obtaining successive solutions of well-posed forward heat conduction problems. The advantages of this iterative BEM are the simplicity of the computational scheme and the high accuracy and stability of the solution. Furthermore, it is applicable to any type of boundary conditions.

In this paper, the convergence of the numerical method and the stability of the numerical solution are illustrated for a very severe test example and the

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Nomenclature

$A, B, C, \ldots$	coefficient matrices
$\underline{b}, \underline{v}$	constant vectors
$[a, b]$	the elements x such that $a \leqslant x \leqslant b$
(a, b)	the elements x such that $a < x < b$
$(a, b]$	the elements x such that $a < x \leqslant b$
$[a, b)$	the elements x such that $a \leqslant x < b$
$\{a, b\}$	the elements $x = a$ and $x = b$
e_T	convergence error
E	error used as a stopping condition
F	fundamental solution of the heat equation
f, g	given values of the temperature
H	Heaviside function
k	number of iterations
L^p	space of p-integrable functions
M	number of time steps
N, N_0	numbers of boundary elements
q	given values of the heat flux
Ω	the space domain
$\partial\Omega$	the boundary of the space domain Ω
$\overline{\Omega}$	the closure of Ω (the set of elements belonging to Ω and its boundary $\partial\Omega$)
t	time
t_f	final time
$\tilde{t}_j$	time nodes
T	temperature
T_0	initial temperature
T_f	final temperature
T'	heat flux
u_k	successive approximations for the temperature
x	space variable
$\tilde{x}_i$	space nodes

Greek symbols

ϵ	Gaussian random variable
η	coefficient of the boundary integral equation
∇^2	Laplace operator
v	normal vector to a surface
σ	standard deviation of a Gaussian random variable

Subscripts and superscripts

i, j, l	indices

computational performances and limitations when t_f increases are investigated.

2. Mathematical formulation of the BHCP

For simplicity, we consider only the one-dimensional backward heat conduction problem. Further, no loss of generality takes place if we assume that the spatial domain of the solution is the interval $[0, 1]$. Thus we consider the following problem:

$$\begin{aligned}
&\frac{\partial T}{\partial t}(x,t) = \frac{\partial^2 T}{\partial x^2} \quad \text{for } (x,t) \in (0,1) \times (0,t_f),\\
&T(0,t) = f_0(t) \quad \text{for } t \in [0,t_f),\\
&T(1,t) = f_1(t) \quad \text{for } t \in [0,t_f),\\
&T(x,t_f) = g(x) \quad \text{for } x \in [0,1],
\end{aligned} \tag{1}$$

where f_0, f_1 and g are prescribed functions and the value of $t_f > 0$ is given. The boundary temperatures f_0, f_1 and the final temperature g are known while the initial temperature $T(\cdot, 0)$ is unknown and has to be retrieved. This is an example of an ill-posed problem which is impossible to solve using classical numerical methods and requires special techniques to be employed. A similar problem is obtained if instead of the temperature, the heat flux is prescribed on the boundaries $x = 0$ and $x = 1$.

For solving the BHCP given by Eq. (1), the quasi-reversibility method of Lattes and Lions [5] constructs an approximation T_ϵ which is the solution of the problem

$$-(\nabla^2 T_\epsilon + \epsilon \nabla^4 T_\epsilon) + \frac{\partial T_\epsilon}{\partial t} = 0 \quad \text{in } \Omega \times (0, t_f), \tag{2}$$

$$T_\epsilon = f, \quad \nabla^2 T_\epsilon = \frac{\partial f}{\partial t} \quad \text{on } \partial\Omega \times [0, t_f], \tag{3}$$

$$T_\epsilon(\cdot, t_f) = g(\cdot) \quad \text{in } \Omega. \tag{4}$$

However, the solution T_ϵ of the well-posed problems (2)–(4) does not converge, in general, when $\epsilon \to 0$. Moreover, in practice, the Dirichlet data f is likely to be contaminated with noise and the differentiation with respect to t in Eq. (3) is in itself an ill-posed problem. The finite-difference method of Lattes and Lions [5] manifested significant instabilities when $t_f > 0.2$. In addition, Eq. (3) cannot be derived if a Neumann boundary condition is prescribed.

Another approach for the problem being considered, is that based on approximating the parabolic BHCP with a convergent sequence of Cauchy ill-posed problems for the approximated elliptic heat equation, which

is then solved by implementing an iterative, alternating algorithm, see [11]. In this approach, an approximation $T_\epsilon, \epsilon > 0$ is constructed, which satisfies

$$L_\epsilon(T_\epsilon) = -\left(\nabla^2 T_\epsilon + \epsilon \frac{\partial^2 T_\epsilon}{\partial t^2}\right) + \frac{\partial T_\epsilon}{\partial t} = 0 \quad \text{in } \Omega \times [0, t_f), \tag{5}$$

$$T_\epsilon = f \quad \text{on } \partial\Omega \times [0, t_f], \tag{6}$$

$$T_\epsilon(\cdot, 0) = T_0(\cdot) = \text{unknown}, \quad T_\epsilon(\cdot, t_f) = g(\cdot) \quad \text{in } \Omega. \tag{7}$$

For very small values of ϵ, the solution T_ϵ of problems (5)–(7) is a good approximation to the solution of the BHCP given by Eq. (1). However, since in Eq. (7), $T_0(\cdot)$ is unknown, problem (5)–(7) is transformed into a Cauchy problem by initially solving the direct, well-posed forward heat conduction problem in the domain $\Omega \times [t_f, +\infty)$ to obtain

$$\frac{\partial T}{\partial v}(\cdot, t_f) = h(\cdot) \quad \text{in } \Omega. \tag{8}$$

The iterative BEM proposed by Lesnic et al. [11] performed reasonable accurate and stable computations up to $t_f = 1.0$. However, in order to accommodate the method of Lesnic et al. [11] it is necessary to know the boundary data for values of t larger than t_f, but these data are not always available in practice. Moreover, whilst the method produces accurate results for small values of t_f the accuracy is substantially decreased as t_f increases.

In this paper, we implement a convergent iterative algorithm proposed by Kozlov and Maz'ya [9] based on an iterative procedure which consists of obtaining successive solutions of well-posed forward heat conduction problems. This algorithm does not need any extra information as required by Lesnic et al. [11].

3. The BEM for the one-dimensional transient heat equation

Following a classical BEM methodology, see [8], and using the fundamental solution for the time dependent heat equation in one dimension given by

$$F(x, t, x', t') = \frac{H(t - t')}{2\sqrt{\pi(t - t')}} \exp(-(x - x')^2/4(t - t')), \tag{9}$$

where H is the Heaviside function, the partial differential Eq. (1) can be transformed into the following boundary integral equation:

$$\eta(x)T(p) = \int_{S_1} \left[F \frac{\partial T}{\partial v} - T \frac{\partial F}{\partial v}\right] \mathrm{d}S_1 + \int_{S_2} TF \, \mathrm{d}S_2, \quad p = (x, t) \in \overline{\Omega} \times (0, t_f), \tag{10}$$

where $S_1 = \{0, 1\} \times (0, t_f)$, $S_2 = \overline{\Omega} \times \{0\}$ and $\eta(x) = 1$ if $x \in (0, 1)$ and $\eta(0) = \eta(1) = 0.5$.

The boundary S_1 is discretised into a series of small boundary elements (t_{j-1}, t_j) for $j = 1, 2, \ldots, N$ whilst the boundary S_2 is discretised into a series of small cells $[x_{k-1}, x_k]$ for $k = 1, 2, \ldots, N_0$. Over each boundary element, the temperature T and the flux $\partial T/\partial v$ are assumed to be constant and take their values at the midpoint $\tilde{t}_j = (t_{j-1} + t_j)/2$. Also, over each cell the temperature T is assumed to be constant and takes its value at the midpoint $\tilde{x}_k = (x_{k-1} + x_k)/2$. If the boundary integral equation (10) has these approximations applied at every node on the boundary S_1, then the following system of linear algebraic equations is obtained, see [8]

$$A\underline{T}' - B\underline{T} + E\underline{T}^0 = 0, \tag{11}$$

where the vectors $\underline{T}', \underline{T}$ and $\underline{T}^0$ contains the values of the heat flux through the boundary, the temperature on the boundary and the initial temperature, respectively. The matrices A, B and E depend solely on the geometry of the solution domain and may be analytically computed.

For the direct problem, the initial temperature $T(\cdot, 0)$ and the boundary temperatures $T(0, \cdot)$ and $T(1, \cdot)$, (or the boundary heat fluxes $(\partial T/\partial v)(0, \cdot)$ and $(\partial T/\partial v)(1, \cdot)$), are known so that the vectors $\underline{T}$, (or $\underline{T}'$), and $\underline{T}^0$ are known and the system given by Eq. (11) has $2N$ linear equations and $2N$ unknowns that can be solved by a direct method, e.g., Gaussian elimination method.

However, for various ill-posed problems the system of linear algebraic equations resulting from a rearrangement of Eq. (11), together with the substitution of the known boundary data and the extra information provided, is ill-conditioned. Thus a direct approach to the problem produces a highly unstable solution and that is why other methods, such that the iterative method presented in the following section, must be developed.

4. Description of the algorithm

The solution of the BHCP may be accurately approximated by a sequence of solutions of well-posed forward heat conduction problems constructed as follows:

Step 1. Specify an initial guess u_0 for the initial temperature $T(\cdot, 0)$.

Step 2. If u_k has been constructed, then solve the well-posed forward heat conduction problem

$$\begin{aligned}
&\frac{\partial T^{(k)}}{\partial t}(x,t) = \frac{\partial^2 T^{(k)}}{\partial x^2}(x,t) \quad \text{for } (x,t) \in (0,1)\times(0,t_f),\\
&T^{(k)}(0,t) = f_0(t) \quad \text{for } t \in [0,t_f),\\
&T^{(k)}(1,t) = f_1(t) \quad \text{for } t \in [0,t_f),\\
&T^{(k)}(x,0) = u_k(x) \quad \text{for } x \in [0,1]
\end{aligned} \tag{12}$$

to determine the kth approximation $T^{(k)}$ for the temperature distribution inside the solution domain.

Step 3. Construct u_{k+1} as given by the equation

$$u_{k+1}(x) = u_k(x) - \gamma(T^{(k)}(x,t_f) - g(x)), \tag{13}$$

where γ is a positive parameter.

Step 4. Repeat steps 2 and 3 until a prescribed stopping criterion is satisfied.

It has been proved, see [9], that the sequence $\{T^{(k)}(x,t))\}_{k\geq 0}$ is convergent to the exact solution of problem (1) for any initial guess u_0 and any sufficiently small parameter γ.

In this paper, we investigate the numerical convergence and the stability of the iterative algorithm described above. We note that by this algorithm, the ill-posed BHCP is reduced to a sequence of well-posed forward heat conduction problems. The intermediate forward problems are solved using a classical BEM as described in the previous section. Therefore, the purpose of this paper is to introduce an iterative BEM numerical implementation of the algorithm of Kozlov and Maz'ya [9] and to investigate the numerical convergence, stability and accuracy with respect to the mesh size discretisation and the number of iterations.

We note that relaxation procedures can also be constructed by relaxing the marching condition (13) but using these relaxation procedures with various constant relaxation factors has the same effect as using various values for the parameter γ. The optimum value for the parameter γ is also investigated in this paper and a criterion for selecting γ is given.

The algorithm can easily be adopted to the situation when the Neumann boundary data are given on a part of the boundary by modifying accordingly the forward problem (12) in step 2 of the algorithm. The method has a general character and can be extended to a wide range of analogous ill-posed problems. For example, similar algorithms may be developed for the Cauchy problem for the steady-state heat equation or for the two-dimensional BHCP.

5. Numerical results and discussion

In order to investigate the convergence and the stability of the numerical algorithm described in the previous section, we investigate the same typical benchmark test example that was considered by Lesnic et al. [11], namely, the temperature to be retrieved is given by

$$\begin{aligned}
&T(x,t) = \sin(\pi x)\exp(-\pi^2 t),\\
&(x,t) \in [0,1]\times[0,t_f].
\end{aligned} \tag{14}$$

Therefore, the BHCP considered can be written as

$$\begin{aligned}
&\frac{\partial T}{\partial t}(x,t) = \frac{\partial^2 T}{\partial x^2}(x,t) \quad \text{for } (x,t) \in (0,1)\times(0,t_f),\\
&T(0,t) = 0 \quad \text{for } t \in [0,t_f),\\
&T(1,t) = 0 \quad \text{for } t \in [0,t_f),\\
&T(x,t_f) = \sin(\pi x)\exp(-\pi^2 t_f) \quad \text{for } x \in [0,1],
\end{aligned} \tag{15}$$

which is a severe test example since $T(x,\cdot)$ decays rapidly to the steady state, zero-solution, as t increases. Also, for this example, the Dirichlet data $f = 0$ and the information given by $g(x) = \sin(\pi x)\exp(-\pi^2 t_f)$ are very weak, i.e., g is very small if t_f is large. For example, for $t_f = 1, g(x)$ is $O(10^{-4})$ which is almost negligible in comparison with the desired initial temperature to be retrieved, namely, $T_0(x) = \sin(\pi x)$ which is $O(1)$. Clearly, if t_f is chosen sufficiently large, such that the order of g will decrease below any possible (finite) computer machine precision, then the inverse BHCP problem will become uncomputable.

Clearly, for the test example given by Eq. (14), apart from the mathematical ill-posedness of the BHCP, the solution of the inverse problem becomes more difficult to obtain computationally as t_f increases. In this section, numerical results are presented for small, $t_f = 0.1$, moderate, $t_f \in \{0.3, 0.5\}$ and large, $t_f \geqslant 0.75$ values of the final time t_f where the measurements are made.

The test example considered by Eq. (14) provides little information for the inverse problem and therefore enables one to judge the performances and computational limitations of the numerical method on highly ill-posed BHCP formulations.

The numbers of boundary elements used to discretise the space domain, N_0, and the time interval $[0,t_f], N$, were both taken to be $N, N_0 \in \{10, 20, 40\}$.

5.1. Initial guess

An arbitrary function u_0 may be specified as a guess for the initial temperature values but in order to improve the rate of convergence of the iterative procedure, we have chosen a function which ensures the continuity of the temperature at the endpoints of the space interval and which is also linear with respect to the spatial coordinate x.

For the test example considered, the linear guess is given by the constant function

$$u_0(x) = 0, \quad x \in [0,1]. \tag{16}$$

5.2. Convergence error

In order to investigate the convergence of the described algorithm, at every iteration we evaluate the convergence error defined by

$$e_T = ||u_k - T_0||_{L^2(\Omega)}, \tag{17}$$

where u_k is the initial temperature in the domain $\Omega = [0,1]$ retrieved after k iterations and T_0 is the exact value of the initial temperature, given by

$$T_0(x) = T(x,0) = \sin(\pi x). \tag{18}$$

Fig. 1 shows, on a log–log scale, the convergence error e_T given by Eq. (17) as a function of the number of iterations $k = 1, 2, \ldots, 20000$, calculated using a midpoint rectangular rule for the norm $||\cdot||_{L^2[0,1]}$, for various values of the final time $t_f \in \{0.1, 0.3, 0.5\}$. The parameter γ was taken to be $\gamma = 0.05$, but similar results were obtained for various values of this parameter, which act as a relaxation factor. A practical way to choose the optimum value for the parameter γ is also presented later in this paper.

We note that the error e_T decreases up to a specific iteration and after that it starts to increase, see Fig. 1. The number of the iteration at which the minimum in the error e_T occurs increases as the final time t_f increases. Thus, as one would expect, the larger the value of the final time, the larger the number of iterations necessary to obtain an accurate solution. Moreover, the minimum value of the error e_T increases as t_f is increased. Therefore, less accurate solutions are obtained for large values of the final time t_f.

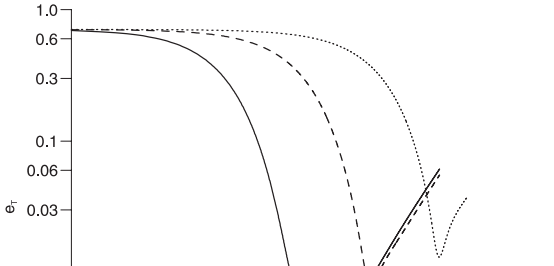

Fig. 1. The error in predicting the initial temperature, e_T, obtained for various values of the final time t_f, namely, $t_f = 0.1$ (–), $t_f = 0.3$ (– – –) and $t_f = 0.5$ ($\cdots$), using $N = 20$ and $N_0 = 40$ boundary elements and $\gamma = 0.05$ for the BHCP given by Eq. (15).

Since the error in predicting the initial temperature T_0 starts increasing after a certain number of iterations, a stopping criterion is required in order to cease the iterations at an optimum point. A stopping criterion, similar to the one based on a direct BEM solution, used for the Cauchy problem for the steady-state heat equation in our previous work, see [11].

5.3. Stopping criterion

As described in Section 3, by applying a classical direct BEM, the following system of linear algebraic equations:

$$A_1\underline{T}' - B_1\underline{T} + E_1\underline{T}^0 = 0 \tag{19}$$

is obtained and thus the direct problem may be accurately solved by replacing the known boundary data in the system and using a Gaussian elimination method.

However, for the BHCP the vector $\underline{T}^0$ is considered unknown and thus the system of linear algebraic equations given by Eq. (11) has $2N + N_0$ unknowns and only $2N$ equations. Nevertheless, N_0 more equations can be obtained if the boundary integral equation (10) is applied at the points $(\tilde{x}_k, t_f)_{k=1,\ldots,N_0}$ using the given final temperature $T(\cdot, t_f)$. These N_0 more equations may be written as

$$A_2\underline{T}' - B_2\underline{T} + E_2\underline{T}^0 = \underline{T}_f, \tag{20}$$

where the vector $\underline{T}_f$ contains the values of the unknown temperature at the final time, $\underline{T} = \left(T(\tilde{x}_j, t_f)\right)_{j=1,\ldots,N_0}$. A new system of $2N + N_0$ linear algebraic equations with $2N + N_0$ unknowns

$$A_1\underline{T}' + E_1\underline{T}^0 = B_1\underline{T}, \quad A_2\underline{T}' + E_2\underline{T}^0 = B_2\underline{T} + \underline{T}_f \tag{21}$$

is obtained and it can be recast as

$$C\underline{X} = \underline{b}, \tag{22}$$

where the vector $\underline{b}$ and the matrix C are known and the vector $\underline{X}$ contains the unknown values of the initial temperature and of the heat flux through the boundary.

The system of linear algebraic equations given by Eq. (22) cannot be solved to produce an accurate solution by a direct approach, since the matrix C, which depend solely on the geometry of the domain and the boundary condition formulation, is ill-conditioned. The condition number of the sensitivity matrix $\text{cond}(C) = \det(C^{\text{tr}}C)$ obtained using various numbers of boundary elements, namely, $N \in \{5, 10, 20\}$ for $t_f = 0.1$ was found to be of the order $10^{-219}, 10^{-241}$ and 10^{-286}, respectively, revealing a high degree of ill-posedness. Thus a direct approach to the problem produces a highly unstable solution and that is why other methods, such that the iterative method presented in this paper must be used.

As for the Cauchy problem for the Laplace equation considered in [12], we consider the system of linear

algebraic equations given by Eq. (22), and we evaluate at every iteration the error

$$E = ||C\underline{X}_k - \underline{b}||, \tag{23}$$

where $\underline{X}_k$ is the vector obtained from the values of the initial temperature and of the heat flux after k iterations and $||\underline{v}||$ denotes the Euclidean norm of the vector $\underline{v}$. The error E should tend to zero as the vector $\underline{X}_k$ tends to the exact solution. Therefore, the error E should provide an appropriate stopping criterion since it contains information about all the unknown data.

In order to investigate the evolution of the error E, Fig. 2 shows on a log–log scale the errors e_T and E as functions of the number of iterations k, for various values of the final time $t_f \in \{0.1, 0.3, 0.5\}$. It can be seen for all the examples considered that at the same time as the error e_T attains its minimum, the error E stops decreasing and tends to become constant. Numerous other test examples have been investigated and the same conclusion may be drawn for various numbers of boundary elements and various values of the parameter γ.

The numerical solution obtained for the initial temperature $T_0 = T(\cdot, 0)$ for various values of the final time $t_f \in \{0.1, 0.3, 0.5\}$ using an iterative BEM with $N_0 = 40$ and $N = 20$ is presented in Fig. 3 and it is found to be in good agreement with the analytical solution given by Eq. (18).

Thus it can be concluded that the described stopping criterion, based on locating the point where the value of the error E stops decreasing and tends to be a constant, generally provides good stable estimates for the exact solution and can be used to stop the iterative process very close to its optimum solution.

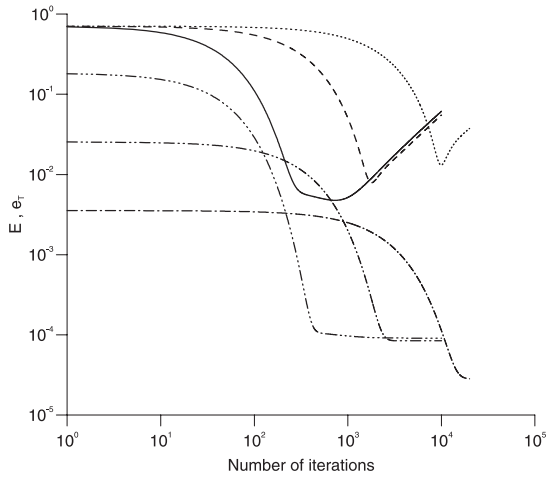

Fig. 2. The errors E obtained for various value of the final time, namely, $t_f = 0.1$ (– · · · –), $t_f = 0.3$ (– · –) and $t_f = 0.5$ (– · – ·) in comparison with the error e_T for the same values of t_f, namely, $t_f = 0.1$ (–), $t_f = 0.3$ (– – –) and $t_f = 0.5$ (· · · ·) obtained using $N = 20$ and $N_0 = 20$ boundary elements and $\gamma = 0.05$, for the BHCP given by Eq. (15).

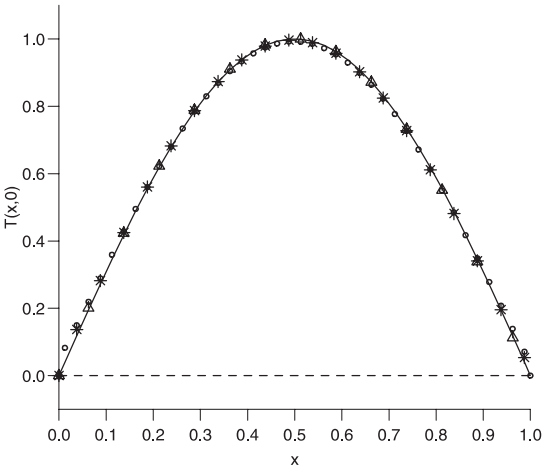

Fig. 3. The numerical solution obtained for the initial temperature $T(\cdot, 0)$ for various values of the final time, namely, $t_f = 0.1$ (△), $t_f = 0.3$ (∗) and $t_f = 0.5$ (○) obtained using an iterative BEM with $N = 20$ and $N_0 = 40$ boundary elements and $\gamma = 0.05$, in comparison with the analytical solution (–) and the initial guess (– – –), for the BHCP given by Eq. (15).

It should be noted that the time and computer storage necessary to obtain a good solution are substantially decreased if the solution in the domain $(0, 1) \times (0, t_f)$ is not computed after every iteration but only at the final iteration after the stopping criterion has been satisfied, since only the boundary data are used to evaluate the stopping error E and to commence a new iteration.

5.4. An alternative way to compute the error E

As in the previous section, we assume that the vectors $\underline{T}$ and $\underline{T}_f$ contain the known values of the temperature on the boundary and at the final time t_f, respectively. We also assume that the vectors $(\underline{T}')^k$ and $(\underline{T}^0)^k$ and $\underline{T}_f^k$ contain the values of the heat flux through the boundary, the initial temperature $T(\cdot, 0)$ and the final temperature $T(\cdot, t_f)$ obtained after k iterations. As described in Section 3, by applying a classical BEM, the following system of linear algebraic equations is obtained at every iteration:

$$A_1(\underline{T}')^k + E_1(\underline{T}^0)^k = B_1\underline{T},$$
$$A_2(\underline{T}')^k + E_2(\underline{T}^0)^k = B_2\underline{T} + \underline{T}_f^k, \tag{24}$$

where the matrices A_i, B_i and E_i, $i = 1, 2$ depend solely on the geometry of the solution domain. On the other

hand, the error E is evaluated as the norm of the residual of the system obtained by applying a direct BEM

$$E = \|C\underline{X}_k - \underline{b}\|, \tag{25}$$

where

$$C = \begin{bmatrix} A_1 & E_1 \\ A_2 & E_2 \end{bmatrix}, \quad b = \begin{bmatrix} B_1\underline{T} \\ B_2\underline{T} + \underline{T}_f \end{bmatrix},$$
$$X_k = \begin{bmatrix} (\underline{T}')^k \\ (\underline{T}^0)^k \end{bmatrix}. \tag{26}$$

The residual $CX_k - b$ may be evaluated as

$$CX_k - b = \begin{bmatrix} A_1(\underline{T}')^k + E_1(\underline{T}^0)^k - B_1\underline{T} \\ A_2(\underline{T}')^k + E_2(\underline{T}^0)^k - B_2\underline{T} - \underline{T}_f \end{bmatrix} \tag{27}$$

and using Eq. (24), we obtain

$$CX_k - b = \begin{bmatrix} 0 \\ \underline{T}_f^k - \underline{T}_f \end{bmatrix}. \tag{28}$$

Thus the error E is given by

$$E^2 = \left\|\underline{T}_f^k - \underline{T}_f\right\|^2 = \sum_{j=1}^{N_0}\left[T^k(\tilde{x}_j, t_f) - g(\tilde{x}_j)\right]^2 \tag{29}$$

with the approximations assumed by the BEM the L^2 norm of the difference between the given final temperature and the final temperature obtained after k iterations may be evaluated as

$$\left\|T^k(\cdot, t_f) - g(\cdot)\right\|_{L^2}^2 = \int_0^1 \left[T^k(x, t_f) - g(x)\right]^2 \mathrm{d}x \tag{30}$$
$$= \sum_{j=1}^{N_0}\int_{x_{j-1}}^{x_j}\left[T^k(x, t_f) - g(x)\right]^2 \mathrm{d}x \tag{31}$$
$$= \sum_{j=1}^{N_0}\left[T^k(\tilde{x}_j, t_f) - g(\tilde{x}_j)\right]^2 (x_j - x_{j-1}) \tag{32}$$
$$= \frac{1}{N_0}\sum_{j=1}^{N_0}\left[T^k(\tilde{x}_j, t_f) - g(\tilde{x}_j)\right]^2 \tag{33}$$
$$= \frac{1}{N_0}E^2. \tag{34}$$

If Eq. (13) is now used, we obtain

$$E = \sqrt{N_0}\left\|T^k(\cdot, t_f) - g(\cdot)\right\|_{L^2} = \frac{\sqrt{N_0}}{\gamma}\|u_{k+1} - u_k\|_{L^2}, \tag{35}$$

i.e., the error E is proportional to the L^2 norm of the difference between two consecutive approximations for the initial temperature $T(\cdot, 0)$. It should be noted that the factor of proportionality $\sqrt{N_0}/\gamma$ depends on the mesh size discretisation and on the relaxation parameter used.

Using Eq. (35), the error E may be evaluated in a simple way using only two successive approximations of the initial temperature or the given final temperature and its corresponding approximation. Thus computational time and storage is saved. It should be noted that a relation similar to (35) does not hold for the error E for the Cauchy problem for Laplace equation considered in [12]. Thus, it can be concluded that the error E obtained by evaluating the norm of the residual of the system generated by a direct BEM is a more general stopping criterion. However, for the algorithm presented in this section this stopping criterion is reduced to the simple, well-known Cauchy criterion, which consists of evaluating the difference between two consecutive terms of the sequence that approximates the unknown function.

5.5. Convergence with respect to the number of boundary elements

In order to investigate the convergence of the method with respect to the number of boundary elements used for the time discretisation, Fig. 4 shows on a log–log scale the curves obtained by plotting the error e_T as a function of the number of iterations, for various values of $N \in \{10, 20, 40\}$ and a fixed $N_0 = 40$. It can be seen that the larger the number of time boundary elements, the smaller is the minimum in the error e_T and thus the more accurate is the numerical solution obtained.

The same conclusion may be drawn if the number of boundary elements in time N is constant and the number of boundary elements in space N_0 is increased. It can be concluded that the iterative process described is convergent with respect to increasing the number of boundary elements used to discretise both the space and time domains. It was found for small values of the final

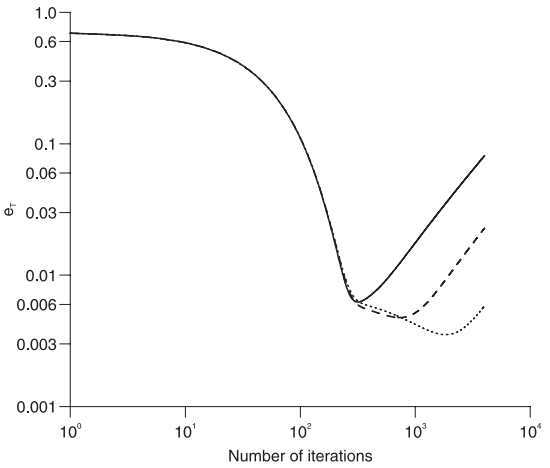

Fig. 4. The error e_T obtained for $t_f = 0.1$ for various numbers of time boundary elements, namely, $N = 10$ (–), $N = 20$ (– – –) and $N = 40$ (· · · ·), and a constant number of space boundary elements $N_0 = 40$, for $\gamma = 0.05$, for the BHCP given by Eq. (15).

time t_f that the accuracy of the numerical solution obtained is not substantially increased by taking N larger than 20. However, for larger values of the final time t_f, increasing values of N may be necessary.

5.6. A time marching scheme

The described algorithm can be employed in a backward time marching scheme by replacing the global interval $[0, t_f]$ with subintervals $[t_i, t_{i+1}]$, starting from $t_f = t_M > t_{M-1} > \cdots > t_1 > t_0 = 0$, where M is the number of time steps.

In what follows, the final time was taken to be $t_f = 0.5$ and the interval $[0, t_f]$ was discretised into M equal subintervals $[t_i, t_{i+1}]$ for $i = 0, M-1$, of length $\Delta t = t_f/M$. Further, each of the subintervals $[t_i, t_{i+1}]$ was discretised using $N = 20$ boundary elements. Over each layer $\Omega \times (t_i, t_{i+1})$ for $i = 0, M-1$ the algorithm previously described was numerically implemented.

The numerical solutions for the initial temperature retrieved using a time marching scheme with various numbers of time steps, $M \in \{1, 2, 5\}$, are graphically represented in Fig. 5 in comparison with the analytical solution. It can be seen that the results are still accurate for small numbers of time steps, $M = 2$, but are inaccurate for large number of time steps, $M = 5$. It should be mentioned that the results obtained without the time marching scheme ($M = 1$) are better than those obtained by marching in time, even if the time marching procedure using a small number of time steps, $M \in \{2, 3\}$, also provide accurate solutions.

5.7. Numerical results for large values of t_f

In order to investigate the performance and the computational limitations of the numerical method described, Fig. 6 shows the numerical solution for the initial temperature obtained for large values of the final time, namely $t_f \in \{0.75, 1.0\}$. It can be seen that the accuracy of the numerical solution decreases as t_f increases, but even for large values of the final time the numerical solution is still a good approximation to the exact solution.

One disadvantage of using the described method for large time intervals is the slow rate of convergence and thus the large number of iterations necessary to obtain an accurate solution. Thus, for the test example given by Eq. (14), the numbers of iterations used to obtain the numerical solutions presented in Fig. 6 are $O(10^6)$. The rate of convergence may be improved by increasing the value of the parameter γ, but too large values for this parameter may result in a loss of accuracy of the final solution, as described in the following section. The large number of iterations necessary for large values of t_f are substantially reduced if a time marching scheme is employed. However, in order to preserve the accuracy of the numerical solution only small numbers of time steps, $M \in \{2, 3\}$, may be used, as mentioned in the previous section.

Fig. 7 presents the numerical solution for the BHCP with $t_f \in \{1.0, 1.5, 2.0\}$ using a time marching scheme with $M = 2$ time steps. It can be seen that there is a good agreement between the numerical and the analytical solutions. Moreover, by using a time marching scheme

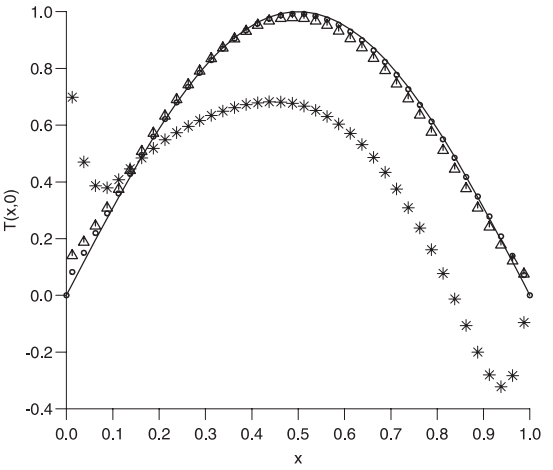

Fig. 5. The numerical solutions obtained for the initial temperature with $t_f = 0.5$ using a marching scheme with various numbers of time steps, namely, $M = 1$ (○), $M = 2$ (△), $M = 5$ (∗), and the analytical solution (–) for the BHCP given by Eq. (15).

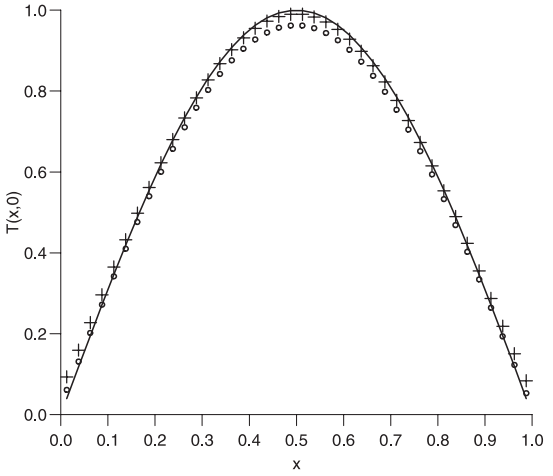

Fig. 6. The numerical solution for the initial temperature obtained using the iterative BEM described over the whole interval $(0, t_f)$ with $\gamma = 0.1$ for various large values of the final time, namely, $t_f = 0.75$ (+) and $t_f = 1.0$ (○), and the analytical solution (–).

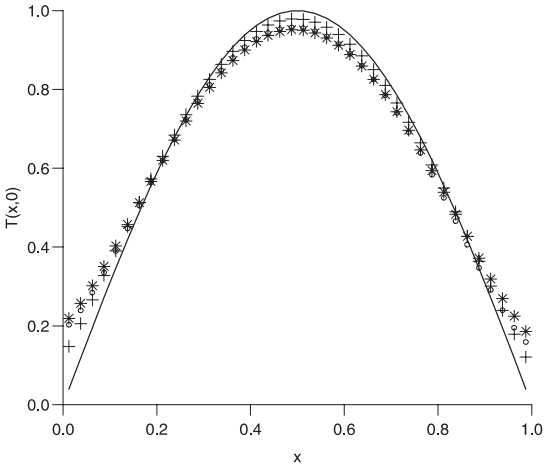

Fig. 7. The numerical solution for the initial temperature obtained using a time marching scheme with $M = 2$ time steps and $\gamma = 0.1$ for various large values of the final time, namely, $t_f = 1.0$ (+), $t_f = 1.5$ (○) and $t_f = 2.0$ (∗) and the analytical solution (–).

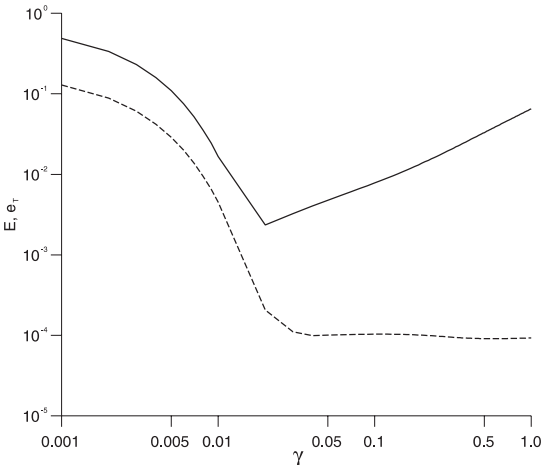

Fig. 8. The errors e_T (–) and E (– – –) as functions of the parameter γ obtained using the described iterative BEM with $N = 20$ and $N_0 = 40$ boundary elements for the BHCP considered by Eq. (15) with $t_f = 0.1$.

with $M = 2$ time steps, the number of iterations necessary to obtain the final solution is reduced by a factor 5.

The largest final time t_f for which the BHCP considered may be accurately solved clearly depends on the test example considered. For the difficult test example given by Eq. (14), the methods proposed by Lattes and Lions [5] and Lesnic et al. [11] manifested significant instabilities when $t_f > 0.2$ and $t_f > 1.0$, respectively. However, the iterative BEM employed in this study was free of such difficulties and we were able to perform reasonably accurate and stable computations up to $t_f = 2.0$.

5.8. Choosing the parameter γ

In order to reduce the number of iterations necessary to obtain an accurate solution, the rate of convergence of the iterative process may be increased by increasing the value of the parameter γ which acts as a constant relaxation factor. However, a large value of the parameter γ may result in a decrease in the accuracy of the final solution. Thus, choosing an appropriate value of the parameter γ becomes important for the accuracy of the iterative procedure. In what follows, a criterion is presented for selecting a value of the parameter γ close to its optimum.

Fig. 8 shows the curves obtained by plotting the errors e_T and E as functions of the parameter γ for $t_f = 0.1$. It is clear from the curve e_T that the optimum value is situated around $\gamma = 0.02$. At the same point the curve E stops decreasing and remains approximately constant. Similar conclusions are obtained for various numbers of boundary elements and various values of the final time t_f. Therefore, the optimum value for γ can be approximately located by using the corner in the curve obtained by plotting E as a function of the parameter γ. Since good results are obtained for a various values of the parameter γ which are close to its optimum value, it can be concluded that the described criterion is efficient in choosing the value of the parameter γ. It should be mentioned that the optimum value of the parameter γ depends on both the test example considered and the value of the final time t_f.

5.9. Stability of the algorithm

In order to investigate the stability of the iterative BEM proposed, the final data $g(\cdot) = T(\cdot, t_f)$ was perturbed as

$$g = g + \varepsilon, \quad \varepsilon = \mathrm{G05DDF}(0, \sigma), \qquad \sigma = \max|g| \frac{s}{100}, \tag{36}$$

where ε is a Gaussian random variable with mean zero and standard deviation σ, generated by the NAG routine G05DDF, see [6], and $s\%$ is the percentage of additive noise included in the input data $g(\cdot) = T(\cdot, t_f)$ in order to simulate the inherent measurement errors.

Fig. 9 shows the numerically obtained initial temperature for $t_f = 0.1$ when various amounts of noise $s \in \{0.1, 0.5, 1.0\}$ are added into the input data. It can be seen that as s decreases the numerical solution approximates better the exact solution given by Eq. (14), while remaining stable. It should be noted that the results are improved if the perturbed input data are smoothed before being used in the iterative algorithm.

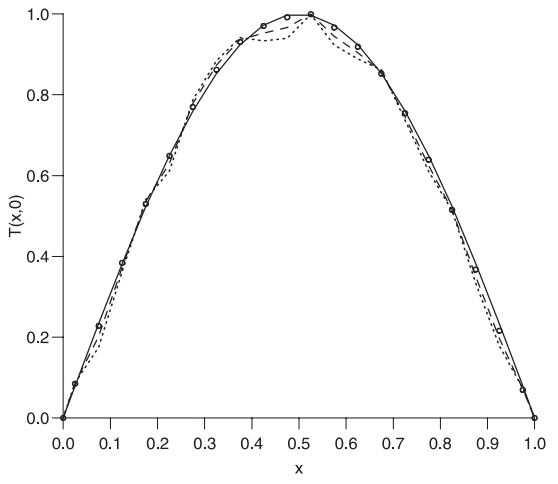

Fig. 9. The numerical solution obtained for the initial temperature for $t_f = 0.1$ with $\gamma = 0.05$ for various amounts of noise added into the input data, namely $s = 0.1\%$ ($\circ\circ\circ$), $s = 0.5\%$ ($- - -$) and $s = 1\%$ ($\cdots$) in comparison with the analytical solution (–), for the BHCP given by Eq. (15).

We note that in order to preserve the stability of the solution it is necessary to use a stopping criterion as described above since the numerical solution produced is not convergent with respect to increasing the number of iterations as it stops improving and the errors start increasing after a certain number of iterations. It was found that the described stopping criterion is efficient in terminating the iterative process close to its optimum point, even if noisy data are used for the final temperature g. Overall, it can be concluded that the proposed iterative BEM produces a convergent, stable and consistent numerical solution with respect to decreasing the amount of noise.

6. Conclusions

In this paper, we have illustrated the use of an iterative BEM for the solution of the one-dimensional backward heat-conduction problem. In order to deal with the instabilities of this ill-posed problem, the BHCP was reduced to a sequence of well-posed forward heat conduction problems. The convergence of the numerical method and the stability of the numerical solution were illustrated for a very severe test example. In addition some computational performances and limitations when t_f increases were discussed. The algorithm was found to be very efficient in retrieving the temperature history even if a large value is used for the final time at which the temperature is prescribed. A stopping criterion and a criterion to chose the appropriate relaxation parameter were also proposed.

Overall, it can be concluded that the iterative BEM proposed produces a convergent, stable and accurate numerical solution with respect to increasing the number of boundary elements and decreasing the amount of noise added into the input data.

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